Comment on "Analytical investigation of the open boundary conditions in the Nagel-Schreckenberg model"

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In their recent paper [Phys. Rev. E **79**, 031115 (2009)], Ning Jia and Shoufeng Ma used some Markov chain arguments for the analytical description of inflow in the deterministic Nagel-Schreckenberg model with open boundaries. In this context, they considered two different mechanisms of injecting vehicles: the standard injection rule and a popular expanded injection rule. While the results for the first one seem to be correct, simulations show that the inflow formula in case of the expanded injection rule yields only approximate results. Therefore, this comment provides the exact formula also in this case and explains the shortcoming in the derivation of Jia and Ma.

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As Jia and Ma wrote [1], cellular automata play an important role in traffic modeling. In this context, the Nagel-Schreckenberg (NS) model [2] is one of the most popular models. Originally developed for the description of freeway traffic, nowadays it is used for modeling urban traffic as well [3]. Typically, it consists of an one-dimensional lattice with L sites (numbered from 1 to L from the left to the right) where each site can either be empty or occupied by a single vehicle. The dynamics are described by three rules which are applied to all vehicles simultaneously at each time step:

$$v_i\left(t+\tfrac{1}{2}\right)=\min\{v_i(t)+1,x_{i+1}(t)-x_i(t)-1,v_{\max}\}, \quad (1)$$

$$v_i(t+1)=\begin{cases}\max\left\{v_i\left(t+\tfrac{1}{2}\right)-1,0\right\} & \text{with prob. } p\\ v_i\left(t+\tfrac{1}{2}\right) & \text{else,}\end{cases} \quad (2)$$

$$x_i(t+1)=x_i(t)+v_i(t+1). \quad (3)$$

Here, $x_i(t)$ and $v_i(t)$ are position and speed of the ith vehicle at time t. The other parameters are the maximum velocity $v_{\max}$ (sites per time step) and the so-called slowdown probability p which causes typical stochastic fluctuations in traffic flow and makes the model more realistic. However, it also makes the analytical investigation of the model much more complex. Because of that, this comment as well as the annotated paper [1] focus on the deterministic case where $p=0$.

Nevertheless, even in the deterministic situation, there are still very interesting phenomena such as phase transitions between free flow and jammed traffic, especially when open boundaries are considered. Furthermore, the density profiles on the lattice are strongly affected by inflow and outflow in this case, too [4,5]. Recently, also the effect of traffic light boundary conditions as a special variant of open boundaries was considered [6,7]. As a result, the generic structure of traffic light queues could be analyzed.

However, in all of these cases, it is crucial to know about the exact inflow which vitally depends on both the injection rate α as well as on the used inflow mechanism. The most common injection rules in this context are the standard injection rule and the expanded injection rule as proposed in [8]. Both are considered in the annotated paper [1].

There, the standard injection rule for each time step is defined as follows [1]: "with probability α a car with velocity $v=v_{\max}$ is created at site 0; this car immediately moves according to the NS rules [i.e., rules (1)–(3)]. If site 1 is occupied by another car, the injected car is deleted."

This injection rule however (cf. [1]) gives rise to some curious inflow behavior which is due to "injection-produced slowdown" (IPSD). Especially, inflow is not necessarily a monotone increasing function of the injection rate α in this case. However, if the left boundary is expanded to a minisystem consisting of $v_{\max}+1$ sites (see [1,8]), the effect of IPSD can be eliminated and the state of maximum flow can be reached which typically is not possible in case of the standard injection rule. Consequently, the expanded injection rule has become very popular during the last years.

In detail, it is defined as follows [8]: "the allocation of the minisystem (left boundary) has to be updated every time step before the vehicles of the complete system. The update procedure consists of two steps. If one cell of the minisystem is occupied, it has to be emptied first. Then a vehicle with initial velocity $v_{\max}$ is inserted with probability q_{in} [i.e., α]. Its position has to satisfy the following conditions: (i) the headway to the first car in the main system is at least equal to the maximum velocity $v_{\max}$ and (ii) the distance to the main system has to be minimal, i.e., if no vehicle is present in the main system within the first $v_{\max}$ cells, the first cell of the boundary is occupied."

Assuming that vehicles are always entering free-flow traffic, now it is interesting to know about the exact inflow $Q_{\text{in}}(\alpha)$ given injection rate α and maximum velocity $v_{\max}$. This is not trivial since vehicles may be hindered to be inserted in some special situations which are generated by the stochastics of the inflow process. Especially, one gets $Q_{\text{in}}(\alpha)\neq\alpha$ for all $\alpha>0$ and $v_{\max}\in\mathbb{N}$.

In [1], some formulas for $Q_{in}(\alpha)$ are presented in case of the standard injection rule as well as for the expanded injection rule. In this context, the results for the standard injection rule agree with simulations very well as shown in several

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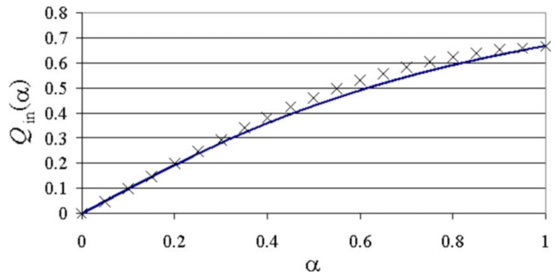

FIG. 1. (Color online) Comparison between simulation results (×) and inflow formula (4) [solid line (blue)] in case of $v_{\max}=2$ for the expanded injection rule.

figures in the same paper [1]. Furthermore, for the expanded injection rule, the paper yields the same formula as presented by Barlovic *et al.* [8] but using a different technique for the derivation:

$$Q_{\text{in}}(\alpha)=\frac{\alpha-\alpha^{v_{\max}+1}}{1-\alpha^{v_{\max}+1}}. \tag{4}$$

Unfortunately, this expression is just approximately correct as recent simulations showed (see Fig. 1).

Seemingly, in both derivations [1,8], the implicit assumption is unintendedly brought in that the further injection behavior does not depend on whether the minisystem at the left boundary is empty or whether its rightmost site is occupied after the injection phase of a given time step. In fact, of course, there is a difference between these two cases (cf. Fig. 2).

For example, let there be a vehicle to be injected. Then, if the rightmost site of the minisystem is occupied after the previous injection phase [cf. Fig. 2(a)], the vehicle at this site first moves to site $2v_{\max}+1$ and thus the new vehicle has to be injected at site $v_{\max}$. If, however, the minisystem is empty after the previous injection phase, i.e., no car has been injected at the previous time step and no car has been injected at site 1 two time steps before [cf. Fig. 2(b)], then the leftmost car will be located on site $2v_{\max}+2$ or to the right of this at the beginning of the current injection phase. Hence, of course, the new vehicle is inserted at site $v_{\max}+1$.

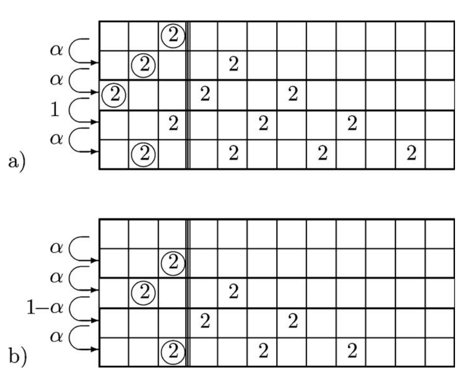

FIG. 2. Difference concerning inflow behavior in case of an empty and nonempty minisystem ($v_{\max}=2$, system after injection phase, new vehicles indicated by circles).

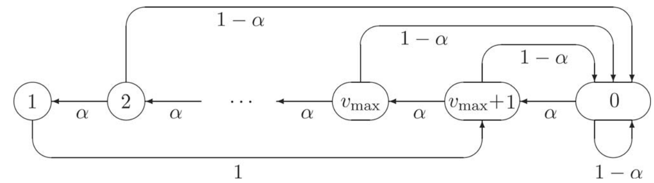

FIG. 3. Transition graph of the Markov chain representing the complete inflow process (based on [9]).

Nevertheless, the Markov chain approach presented in [1] can be adopted also without the above assumption. For this purpose, the transition graph of the Markov chain representing the inflow process has to be just slightly modified (cf. [9]).

For, let $S:=\{0,1,\ldots,v_{\max}+1\}$ be the state space of this modified Markov chain. Then, state 0 describes the situation where the whole minisystem at the left boundary is empty after the injection phase (before updating the complete system). Furthermore, similar to [1], the other states $1,\ldots,v_{\max}+1$ each represent that the corresponding site of the minisystem is occupied by a vehicle at the end of the injection phase. Of course, this vehicle may have been injected at any time step before as is possible in case of state $v_{\max}+1$.

Finally, the corresponding transition graph looks like in Fig. 3 (see also [9]). Obviously, it is very similar to the related transition graph constructed by Jia and Ma except for the additional state 0 which replaces the state $v_{\max}+1$ in a certain way (cf. [1]).

As can be seen easily from Fig. 3, the modified Markov chain is ergodic for all $\alpha\in(0,1)$. Hence, there is an equilibrium distribution $P:=(p_0,\ldots,p_{v_{\max}+1})$ with stationary probabilities p_i for the states $i=0,1,\ldots,v_{\max}+1$. It can directly be obtained by solving the linear equation

$$P=PT \tag{5}$$

with $\sum_{i=0}^{v_{\max}+1}p_i=1$, where T is the one-step transition matrix of the considered Markov chain.

Then, as mentioned in [1] (see also [9]), the correct inflow can be computed by

$$Q_{\text{in}}(\alpha)=\alpha(1-p_1). \tag{6}$$

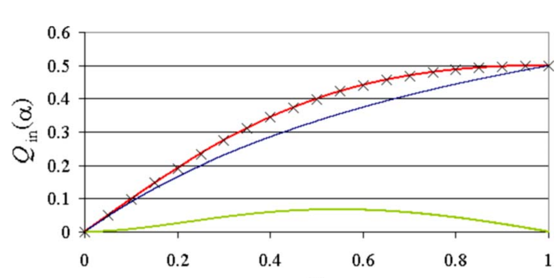

FIG. 4. (Color online) Comparison between simulation results (×) and inflow formulas (4) [middle line (blue)] and (8) [upper line (red)] in case of $v_{\max}=1$ for the expanded injection rule. The lower line (green) shows the difference between both formulas.

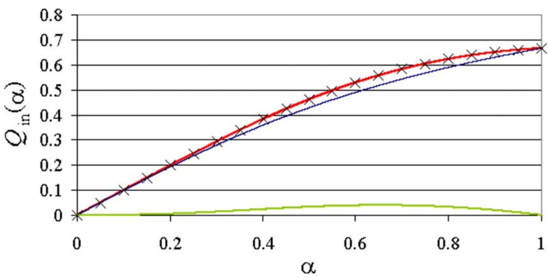

FIG. 5. (Color online) Comparison between simulation results (×) and inflow formulas (4) [middle line (blue)] and (8) [upper line (red)] in case of $v_{max}=2$ for the expanded injection rule. The lower line (green) shows the difference between both formulas.

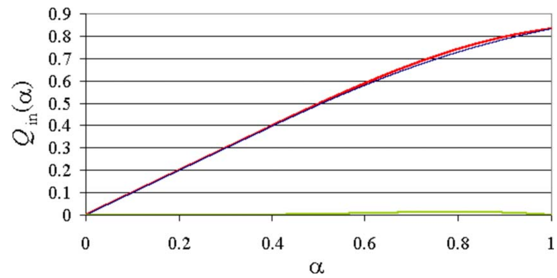

FIG. 6. (Color online) Comparison between inflow formulas (4) [middle line (blue)] and (8) [upper line (red)] in case of $v_{max}=5$ for the expanded injection rule. The lower line (green) shows the difference between both formulas.

Hence, only the stationary probability p_1 of state 1 is needed which is given by (see [9] for details)

$$p_1 = \frac{\alpha^{v_{max}+1}(1-\alpha)}{1-\alpha^{v_{max}}+\alpha^{v_{max}+1}-\alpha^{v_{max}+2}}. \tag{7}$$

Inserting this into Eq. (6) finally yields the correct inflow expression

$$Q_{in}(\alpha) = \frac{\alpha(1-\alpha^{v_{max}})}{1-\alpha^{v_{max}}+\alpha^{v_{max}+1}-\alpha^{v_{max}+2}}. \tag{8}$$

Simulations show Eq. (8) to be exact. Figures 4 and 5 depict the results in comparison to the formula [see Eq. (4)] proposed in [1]. Accordingly, the difference between both analytical expressions of inflow can be seen, too.

However, the deviation between formulas (4) and (8) becomes smaller and smaller as v_{max} increases (cf. Fig. 6). Moreover, both expressions approach the identity function $Q_{in}^{\infty}(\alpha)=\alpha$ as $v_{max}\to\infty$. Since the error is very small already at $v_{max}=5$ (see Fig. 6), this might be the reason why it has not been recognized before that formula (4) is not exact. Vice versa, the maximum deviation appears when $v_{max}=1$. Based on some standard algebra, it can be found that this maximum difference happens at $\alpha\approx 0.543\,689$ with $\Delta Q_{in}\approx 0.067\,442$ in this case (cf. Fig. 4).

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